1 Linear Elliptic PDEs

Let $\Omega \subseteq \mathbb{R}^n$ be an open and bounded subset. A linear partial differential equation of second order is an equation of one of the following forms:

$$a^{ij}(x)\partial_{ij}u(x) + b^i(x)\partial_i u(x) + c(x)u(x) = f(x) \tag{1}$$

or

$$(a^{ij}(x)\partial_i u(x))_j + b^i(x)\partial_i u(x) + c(x)u(x) = f(x) \tag{2}$$

for given functions $a_{ij}, b_i, c, f \ (i, j = 1, \ldots, n)$ defined on $\Omega \subseteq \mathbb{R}^n$. We say that (1) is an equation in *nondivergence form*, whereas (2) is in *divergence form*. Notice that if the coefficients $a_{ij}(x)$ are C^1 functions then the two forms are equivalent.

Remark. We shall assume throughout the text that $a_{ij}(x) = a_{ji}(x)$.

The lefthand side of either equation defines a partial differential operator L and we can alternatively represent these equations as $Lu = f$. We call L a (linear) partial differential operator of second order.

Definition. The operator L is **elliptic** if the functions $a^{ij}(x)$ define a positive definite matrix $[a^{ij}(x)]$. Alternatively, let $\lambda(x), \Lambda(x)$ be the minimum and maximum eigenvalues of the matrix $[a^{ij}(x)]$ then for every $\xi \neq 0$ we have:

$$0 < \lambda(x)|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda(x)|\xi|^2 \tag{3}$$

If $\lambda(x) \geq \lambda_0 > 0$ for some constant λ_0, L is called **strictly elliptic**. If the quotient Λ/λ is bounded then L is **uniformly elliptic**. Lastly, if either $\lambda = 0$ or $\Lambda = \infty$ then L is **degenerate elliptic**.

Example 1. (Laplace and Poisson's Equations) The most basic elliptic equation we can get is by setting $a^{ij}(x) = \delta_{ij}, b^i, c, f \equiv 0$ in (1). The resulting equation

$$\Delta u(x) = 0 \tag{4}$$

is called Laplace's Equation, where $\Delta := \partial_{11} + \ldots \partial_{nn}$. If we allow nonzero $f(x)$, we get

$$\Delta u(x) = f(x) \tag{5}$$

which is Poisson's equation. Both equations arise in physics usually related to diffusion processes like heat and concentration, or potentials, like gravitational and electrostatic potentials.

Remark. Notice that $\Delta u(x) = \text{div}(\nabla u(x))$, which is Laplace's equation in divergence form.

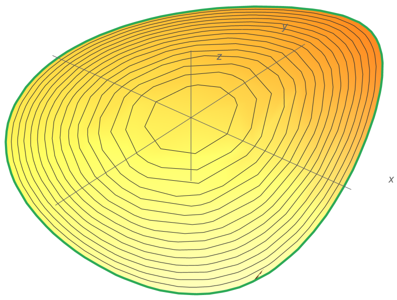

FIGURE 1 Real part of e^z.

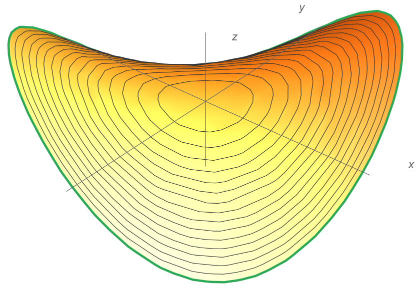

FIGURE 2 Imaginary part of z^2.

2 Harmonic functions

Definition. A **classical solution** $u(x)$ of the elliptic equation $Lu = f$ is a C^2 function $u : \Omega \to \mathbb{R}$ that satisfies the equation pointwise. A function $u(x)$ is called **harmonic** in Ω if it is a classical solution of the Laplace's equation $\Delta u = 0$ in Ω.

Example 2. Let $h : \Omega \subseteq \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Suppose $h(z) = u(x,y)+iv(x,y)$, where $z = x+iy$ and $u, v : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$ are the real and imaginary parts of $h(z)$ respectively. Then the Cauchy-Riemann equations imply that both u and v are harmonic in Ω. For example, take $f(z) = e^z$ then $u(x,y) = \mathrm{Re}(f(z)) = e^x \cos(y)$ is harmonic, see Figure 1 below.

Example 3. Set $\Omega = \mathbb{R}^2 - \{0\}$ and consider the function $u(x,y) = \ln(x^2+y^2)$. One can easily check that u is harmonic in Ω.

Example 4. More generally, in $\mathbb{R}^n$ any constant or linear function in $\Omega \subseteq \mathbb{R}^n$ is harmonic, as it is $u(x) = |x|^{1-n/2}$ for $n > 2$.

Harmonic functions are homogeneous in the sense that it is the average of its values, more precisely we have

Theorem 2.1. *(Mean Value) Let $u \in C^2(\Omega)$. Then $u(x)$ is harmonic if and only if for any ball $B_R(x) \subset\subset \Omega$, we have*

$$u(x) = \fint_{B_R(x)} u(x)\,\mathrm{dx} = \fint_{\partial B_R(x)} u\,\mathrm{ds}. \tag{6}$$

Proof. ($\Rightarrow$) Define $\phi(r) := \fint_{\partial B_r(x)} u(y)\,\mathrm{ds} = \fint_{\partial B_1(0)} u(x+rz)\,\mathrm{ds}$. Taking the derivative we have:

$$\phi'(r) = \fint_{\partial B_1(0)} \nabla u(x+rz)\cdot z\,\mathrm{ds} = \fint_{\partial B_r(x)} \nabla u(y)\cdot \frac{y-x}{r}\,\mathrm{ds} \tag{7}$$

Notice that $\nu = \frac{y-x}{r}$ is normal to $\partial B_r(x)$, by the divergence theorem we have

$$\phi'(r) = \fint_{\partial B_r(x)} \nabla u(y)\cdot\nu\,\mathrm{ds} = \fint_{\partial B_r(x)} \frac{\partial u}{\partial \nu}\,\mathrm{ds} = \frac{r}{n}\fint_{B_r(x)} \Delta u(y)\mathrm{dy} = 0. \tag{8}$$

thus $\phi(r)$ is constant. Letting $r \to 0$ we get

$$\phi(r) = \lim_{r\to 0} \phi(r) = u(x). \tag{9}$$

Additionally, using the above result and the use of polar coordinates

$$\int_{B_r(x)} u(y)\mathrm{dy} = \int_0^\infty \left(\int_{\partial B_t(x)} u \,\mathrm{ds} \right) \mathrm{dt} \tag{10}$$

gives that $⨍_{B_R(x)} u(x)\,\mathrm{dx} = ⨍_{\partial B_R(x)} u\,\mathrm{ds}$.

($\Leftarrow$) The proof is by contradiction. Assume $\Delta u(x) > 0$ for some ball $B_r(x)$, then as before

$$0 = \phi'(r) = \frac{r}{n} ⨍_{B_r(x)} \Delta u(y)\mathrm{dy} > 0 \tag{11}$$

a contradiction. $\square$

The next theorem illustrates a direct consequence of the mean value property. Namely, values of nonnegative harmonic functions are all comparable in a precise sense stated below.

Theorem 2.2. *(Harnack's inequality) Let $u(x)$ be a harmonic function such that $u(x) \geq 0$ for every $x \in \Omega$, and consider a connected subset $U \subset\subset \Omega$. Then there is a constant $C > 0$, depending only on U, such that*

$$\sup_{x\in U} u(x) \leq C \inf_{x\in U} u(x) \tag{12}$$

Proof. It's enough to find $C > 0$ such that

$$\frac{1}{C} u(y) \leq u(x) \leq C u(y) \tag{13}$$

for every $x, y \in U$ and apply the definition of supremum/infimum. The idea is then prove this fact in each ball and use the compactness of $\overline{U}$.

Choose $r > 0$ and $z \in \Omega$ such that $B_{4r}(z) \subset \Omega$ and consider $x, y \in B_{4r}(z)$. By the mean value property we have:

$$\begin{aligned} u(x) &= ⨍_{B_r(x)} u(p)\mathrm{dp} \leq \frac{1}{\omega_n R^n} \int_{B_{2r}(z)} u(p)\mathrm{dp} \\ u(y) &= ⨍_{B_{3r}(y)} u(p)\mathrm{dp} \geq \frac{1}{\omega_n (3R)^n} \int_{B_{2r}(z)} u(p)\mathrm{dp} \end{aligned} \tag{14}$$

It follows that

$$\frac{1}{3^n} u(y) \leq u(x) \leq 3^n u(y), \tag{15}$$

so the result is valid inside balls. But then we are done, since we can find r and cover $\overline{U}$ with finitely many balls $\{B_i\}_{i=1}^N$ of radius r such that $B_i \cap B_{i+1} \neq \emptyset$. Applying the previous estimate in each ball we get

$$\frac{1}{3^{nN}} u(y) \leq u(x) \leq 3^{nN} u(y). \tag{16}$$

Now take $C = 3^{nN}$. $\square$

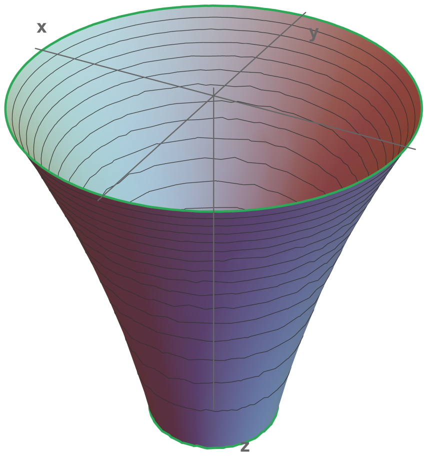

FIGURE 3 $u(x,y) = \ln(x^2 + y^2)$.

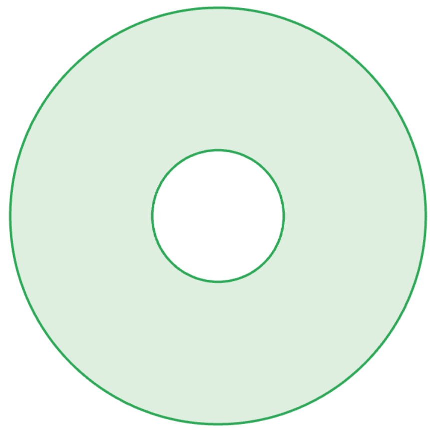

FIGURE 4 The annulus $\frac{1}{e} \leq x^2 + y^2 \leq 1$.

3 Maximum and Minimum Principles

An interesting property of harmonic functions, and more generally elliptic partial differential equations, is the existence of some type of maximum(minimum) principle, which says that the maximum(minimum) is achieved at the boundary.

Theorem 3.1. *(Strong Maximum/Minimum Principle) Let $u(x)$ be harmonic in Ω and continuous on $\overline{\Omega}$. If there's a point $a \in \Omega$ such that $u(a) = \max_{x \in \overline{\Omega}} u(x)$ $(\min_{x \in \overline{\Omega}} u(x))$, then $u(x)$ is constant.*

Proof. The proof is again by contradiction. Suppose the maximum is achieved at a point $a \in \Omega$ and set $M := u(a)$. Consider the non empty set $O := \{x \in \Omega \,|\, u(x) = M\}$. By the continuity of $u(x)$, O is closed. Now, by the mean value theorem, we have:

$$M = u(a) = \fint_{B_R(a)} u(x)\,\mathrm{dx} \leq M \tag{17}$$

for some $R > 0$, the equality holds if and only $u \equiv M$ in $B_R(x)$, hence O is open and since Ω is connected we must have $\Omega = O$. A similar argument can be given in case $u(a) = \min_{x \in \overline{\Omega}} u(x)$. □

A consequence of the max/min principle is a global estimate for harmonic functions. In fact, we'll see this principles generalizes to other type of elliptic equations as well.

Corollary 1. *(Weak Maximum/Minimum Principle) Let $u(x) \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be harmonic in Ω. Then*

$$\max_{x \in \overline{\Omega}} u(x) = \max_{x \in \partial\Omega} u(x) \left(\min_{x \in \overline{\Omega}} u(x) = \min_{x \in \partial\Omega} u(x) \right) \tag{18}$$

so that

$$\min_{x\in\partial\Omega} u(x) \le u(x) \le \max_{x\in\partial\Omega} u(x) \tag{19}$$

Example 5. Consider the harmonic function $u(x,y) = \ln(x^2+y^2)$ over the domain $\Omega = \{(x,y) \in \mathbb{R}^2 \mid \frac{1}{e} < x^2+y^2 < 1\}$. According to the strong max/min principle it should achieve its maximum and minimum values at the boundary of Ω. Indeed, as shown in Figure 3, the maximum value is 0 and the minimum is -1, both achieved at the boundary of the annulus.

The following comparison principle characterizes harmonic functions by their boundary values. Namely, every harmonic function is uniquely defined by its boundary behavior.

Corollary 2. *(Comparison Principle) Let $u(x), v(x) \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be two harmonic functions in Ω such that $u(x) = v(x)$ for $x \in \partial\Omega$. Then $u \equiv v$ in Ω.*

Proof. Set $w := u - v$ and apply Corollary 1 to get $w \equiv 0$ in Ω. □

Next we discuss the *regularity* of harmonic functions. More precisely, given that $u(x) \in C^2$ is harmonic, can we somehow show that $u(x)$ is in fact $C^3(\Omega)$? Or maybe even C^∞? In fact, the latter is true as the following theorem shows (actually the theorem is slightly stronger).

Theorem 3.2. *(Regularity) Any continuous function $u(x) \in C^0$ that satisfies the mean value property 6 is smooth, i.e. $u(x) \in C^\infty$.*

Proof. The proof is by **mollification**, a standard technique in PDEs that proves a result first for smooth functions and then using approximation techniques, proves the result to a broader class of functions. In this particular case, the function itself is already smooth enough, hence coincides with its mollification. This very simple but powerful technique is used in many other PDEs not necessarily elliptic ones, so it is of independent interest. See for example [**?**] for an example of the use of mollifiers to prove local existence of solutions to the Navier-Stokes equations, a parabolic system of equations.

The idea of the proof is 'to mollify' $u(x)$ by taking the convolution of it with a well behaved function called a *mollifier* (See the Appendix). Let $\eta(x) \in C^\infty(\mathbb{R}^n)$ be the standard mollifier and set $u_\epsilon(x) := (u * \eta_\epsilon)(x)$. By the properties of mollifiers, we know that $u_\epsilon \in C^\infty(\Omega_\epsilon)$, we claim that in fact $u \equiv u_\epsilon$ for each $\epsilon > 0$. Take $x \in \Omega_\epsilon$, then:

$$\begin{aligned}
u_\epsilon(x) &= \int_{B_\epsilon(x)} \eta_\epsilon(x-y)u(y)\,\mathrm{dy} \\
&= \frac{1}{\epsilon^n}\int_{B_\epsilon(x)} \eta\left(\frac{|x-y|}{\epsilon}\right) u(y)\,\mathrm{dy} \text{ \textit{(now we switch to polar coordinates)}} \\
&= \frac{1}{\epsilon^n}\int_0^\epsilon \eta\left(\frac{r}{\epsilon}\right)\left(\int_{\partial B_r(x)} u(y)\,\mathrm{ds}\right)\mathrm{dr} \qquad (20) \\
&= u(x)\int_{B_\epsilon(0)} \eta_e(y)\,\mathrm{dy} \\
&= u(x).
\end{aligned}$$

□

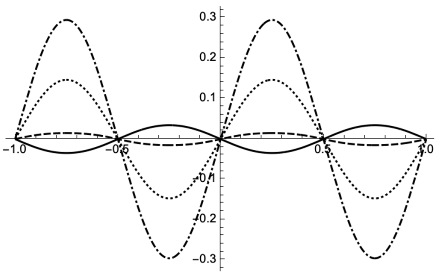

FIGURE 5 $u_\epsilon(x)$ for $\epsilon = 1, .75, .5, .25$ for the sawtooth function

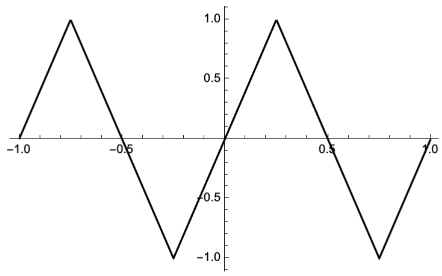

FIGURE 6 The Sawtooth function $u(x) = 4|x - \lfloor x + 3/4 \rfloor + 1/4| - 1$.

The regularity of solutions to partial differential equations is a major topic in the field of differential equations. Most equations that we will discuss in this text actually have at least Holder continuous solutions under certain conditions on the boundary $\partial\Omega$. On the other hand, there are an abundant number of counterexamples to different types of regularites that we will also discuss.

Notice that if $u(x)$ is harmonic for $x \in \Omega$ then $u_i(x)$ is also harmonic for $i = 1, \dots, n$, indeed

$$\Delta u(x) = 0 \iff \partial_i(\Delta u(x)) = 0 \iff \Delta u_i(x) = 0 \tag{21}$$

The next theorem, which is yet another consequence of the mean value theorem, gives an estimate of the derivative locally in terms of the L^1 norm of $u(x)$

Theorem 3.3. *Let $u(x)$ be harmonic in Ω. Then for every ball $B_r(a) \subset \Omega$ and every multiindex α such that $|\alpha| = k$, the following estimate is true*

$$|D^\alpha u(a)| \leq \frac{C_k}{r^{n+k}} \|u\|_{L^1(B_r(a))} \tag{22}$$

with $C_0 = \frac{1}{\omega_n}$ and $C_k = \frac{(2^{n+1}nk)^k}{\omega_n}$

Proof. The proof is by induction on k. The case $k = 0$ is immediate from the mean value theorem. Let $k = 1$, and consider the mean value for the ball $B_{\frac{r}{2}}(a)$:

$$\begin{aligned} |u_i(a)| &= \left| \fint_{B_{r/2}(a)} u_i(y) \,\mathrm{dy} \right| \\ &= \frac{2^n}{\omega_n r^n} \left| \int_{\partial B_{r/2}(a)} u\nu_i \,\mathrm{ds} \right| \\ &\leq \frac{2n}{r} \|u\|_{L^\infty(\partial B_{r/2}(a))} \end{aligned} \tag{23}$$

For every $x \in \partial B_{r/2}(a)$, notice that $B_{r/2}(x) \subset B_r(a)$, hence

$$|u(x)| \leq \frac{2^n}{\omega_n r^n} \|u\|_{L^1(B_r(a))}, \tag{24}$$

which combined with the above inequality gives

$$|D^\alpha u(a)| \leq \frac{2^{n+1} n}{\omega_n r^{n+1}} \|u\|_{L^1(B_r(a))} \tag{25}$$

proving the case $k = 1$. The general case is proved analogously, suppose the result is valid for α with $|\alpha| = k-1$, we prove the result for $|\alpha| = k$. Fix α with $D^\alpha u = (D^\beta u)_i$, with $|\beta| = k-1$. By the computations above, we deduce that

$$|D^\alpha u(a)| \leq \frac{nk}{r} \left\|D^\beta u\right\|_{L^\infty(\partial B_{r/k}(a))} \tag{26}$$

As before, if $x \in \partial B_{r/k}(a)$ then $B_{\frac{(k-1)r}{k}}(x) \subset B_r(a)$ and by hypothesis

$$|D^\beta u(x)| \leq \frac{(2^{n+1} n(k-1))^{k-1}}{\omega_n \left(\frac{k-1}{k} r\right)^{n+k-1}} \|u\|_{L^1(B_r(a))} . \tag{27}$$

Combining everything together we conclude

$$|D^\alpha u(a)| \leq \frac{(2^{n+1} nk)^k}{\omega_n r^{n+k}} \|u\|_{L^1(B_r(a))} . \tag{28}$$

□

As a direct consequence we get an analogous Liouville's theorem for harmonic functions, namely

Theorem 3.4. *(Liouville's theorem) Any bounded $u : \mathbb{R}^n \to \mathbb{R}$ harmonic function is constant.*

Proof. Just notice that if $u(x)$ is bounded then $\|u\|_{L^1(B_r(a))} \leq \|u\|_{L^\infty(\mathbb{R}^n)} < \infty$ and let $r \to \infty$ in the previous theorem with $k = 1$ to conclude that $Du \equiv 0$, hence u is constant. □

We end this section with a stronger version of theorem 3.2.

Theorem 3.5. *If $u(x)$ is harmonic in Ω then $u(x)$ is in fact analytic in Ω.*

Proof. The proof is done using the definition of analyticity. Namely, we must show that given a point $a \in \Omega$, the Taylor series of $u(x)$, $\sum_\alpha \frac{D^\alpha u(a)}{\alpha!}(x-a)^\alpha$, converges to $u(a)$.

Choose r such that $4r = \text{dist}(a, \partial\Omega)$. Then for any $x \in B_r(a)$ we have $B_r(x) \subset B_{2r}(a) \subset \Omega$, moreover by the local derivative estimate 22, and letting $|\alpha| = k$, we have the bound

$$\begin{aligned} \|D^\alpha u\|_{L^\infty(B_r(a))} &\leq \frac{(2^{n+1} nk)^k}{\omega_n r^{n+k}} \|u\|_{L^1(B_{2r}(a))} \\ &\leq \frac{\|u\|_{L^1(B_{2r}(a))}}{\omega_n r^n} \left(\frac{2^{n+1} n}{r}\right)^k k^k \\ &\leq \frac{\|u\|_{L^1(B_{2r}(a))}}{\omega_n r^n} \left(\frac{2^{n+1} n^2 e}{r}\right)^k \alpha! \\ &\leq C \left(\frac{2^{n+1} n^2 e}{r}\right)^k \alpha! \end{aligned} \tag{29}$$

where we have used the fact that $k^k \le e^k k!$ and $k! \le n^k \alpha!$.

Now we claim that the Taylor series $\sum\limits_{\alpha} \frac{D^\alpha u(a)}{\alpha!}(x-a)^\alpha$ converges in the ball $|x-a| \le \frac{r}{2^{n+2}n^2 e}$. Indeed, according to Taylor's theorem, the remainder is given by

$$R_N(x) := u(x) - \sum_{k=0}^{N-1} \sum_{|\alpha|=k} \frac{D^\alpha u(a)}{\alpha!}(x-a)^\alpha = \sum_{|\alpha|=N} \frac{D^\alpha u(a+t(x-a))}{\alpha!}(x-a)^\alpha \tag{30}$$

for some $t \in [0,1]$.

By the discussion above, we must have

$$\begin{aligned} |R_N(x)| &\le C \sum_{|\alpha|=N} \left(\frac{2^{n+1}n^2 e}{r}\right)^N \alpha! \left(\frac{r}{2^{n+2}n^2 e}\right)^N \\ &\le C n^N \frac{1}{(2n)^N} \to 0 \ \text{ as } N \to \infty \end{aligned} \tag{31}$$

□

4 Green's functions

In this section we are interested in obtaining a representation formula for harmonic functions.

We start by recalling the *Green's identities*, which are a consequence of the Divergence theorem in Calculus.

Let $u, v \in C^2(\overline{\Omega})$ and Ω be a bounded domain with C^1 boundary, then:

$$\begin{aligned} &\int_\Omega v\Delta u \,\mathrm{dx} + \int_\Omega Du \cdot Dv \,\mathrm{dx} = \int_{\partial\Omega} v\frac{\partial u}{\partial \nu}\,\mathrm{ds} \qquad \textit{(Green's first identity)} \\ &\int_\Omega (v\Delta u - u\Delta v)\,\mathrm{dx} = \int_{\partial\Omega} \left(v\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}\right)\mathrm{ds} \qquad \textit{(Green's second identity)} \end{aligned} \tag{32}$$

We plan to use the above identities when v is harmonic and u is any other function. In order to accomplish that, we must find an explicit harmonic function. One way we can do this is to find radially symmetric solutions $v(x) = f(|x|)$, that is to say, one dimensional functions depending only on $r = |x|$. Plugging $f(r(x))$ into the Laplace's equation, we can easily obtain the following functions:

$$f(r) = \begin{cases} r^{2-n}, & n > 2 \\ \log(r), & n = 2. \end{cases} \tag{33}$$

Notice that $f(r)$ is singular at 0. It is convenient to shift the singularity to fixed point $y \in \Omega$ and consider $f(|x-y|)$, more precisely, we define the normalized *fundamental solution* of the Laplace's equation to be

$$\Gamma(x-y) = \begin{cases} \frac{1}{n(2-n)\omega_n}|x-y|^{2-n}, & n > 2 \\ \frac{1}{\pi}\log(|x-y|), & n = 2. \end{cases} \tag{34}$$

Since Γ is singular whenever $x = y$, we can't plug it in directly into 32, unless we shrink the domain and consider $\Omega - \overline{B_r(y)}$ instead, for some small $r > 0$. If we do that in Green's second identity, we obtain

$$\int_{\Omega-\overline{B_r(y)}} \Gamma \Delta u \,\mathrm{dx} = \int_{\partial\Omega} \left(\Gamma \frac{\partial u}{\partial \nu} - u \frac{\partial \Gamma}{\partial \nu} \right) \mathrm{ds} + \int_{\partial B_r(y)} \left(\Gamma \frac{\partial u}{\partial \nu} - u \frac{\partial \Gamma}{\partial \nu} \right) \mathrm{ds}. \tag{35}$$

The idea now is to see what happens when we let $r \to 0$.

Lemma 4.1. $\int_{\partial B_r(y)} \left(\Gamma \frac{\partial u}{\partial \nu} - u \frac{\partial \Gamma}{\partial \nu} \right) \mathrm{ds} \to u(y)$ *as* $r \to 0$.

Proof. Notice that

$$\begin{aligned} \int_{\partial B_r(y)} \Gamma \frac{\partial u}{\partial \nu} \mathrm{ds} &= \Gamma(r) \int_{\partial B_r(y)} \frac{\partial u}{\partial \nu} \mathrm{ds} \\ &\leq n\omega_n r^{n-1} \Gamma(r) \sup_{B_r(y)} |Du|, \end{aligned}$$

which goes to 0 as $r \to 0$.

Moreover,

$$\int_{\partial B_r(y)} u \frac{\partial \Gamma}{\partial \nu} \mathrm{ds} = \Gamma'(r) \int_{\partial B_r(y)} u \,\mathrm{ds} = \frac{-1}{n\omega_n r^{n-1}} \int_{\partial B_r(y)} u \,\mathrm{ds}$$

As $r \to 0$ the latter integral goes to $-u(y)$. □

Therefore, when r approaches zero in 35, we obtain a nice formula for any function in terms of the fundamental solution, more precisely we have

Theorem 4.2. *(Green's representation formula) Let* $u \in C^2(\overline{\Omega})$ *and* Ω *be a bounded domain with* C^1 *boundary. For every* $y \in \Omega$*, the following formula holds:*

$$u(y) = \int_{\partial\Omega} \left(u \frac{\partial \Gamma}{\partial \nu} - \Gamma \frac{\partial u}{\partial \nu} \right) \mathrm{ds} + \int_{\Omega} \Gamma(x-y) \Delta u(x) \,\mathrm{dx}. \tag{36}$$

The following corollaries is immediate:

Corollary 3. *Let* $u \in C^2(\overline{\Omega})$ *and* Ω *be a bounded domain with* C^1 *boundary. If* $u(x)$ *has compact support then:*

$$u(y) = \int_{\Omega} \Gamma(x-y) \Delta u(x) \,\mathrm{dx}.$$

If $u(x)$ *is harmonic then*

$$u(y) = \int_{\partial\Omega} \left(u \frac{\partial \Gamma}{\partial \nu}(x-y) - \Gamma(x-y) \frac{\partial u}{\partial \nu} \right) \mathrm{ds}.$$

Now suppose $v(x) \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ is harmonic in Ω. Using Green's second identity 32 we obtain:

$$-\int_{\partial\Omega} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right) \mathrm{ds} = \int_\Omega v\Delta u \, \mathrm{dx}.$$

Set $G(x, y) = \Gamma(x - y) + v(x)$ and suppose $G(x, y) \equiv 0$ for every fixed y. If we add the above expression to 36 we have

$$u(y) = \int_{\partial\Omega} u\frac{\partial G}{\partial \nu} \, \mathrm{ds} + \int_\Omega G\Delta u \, \mathrm{dx}. \tag{37}$$

The function $G(x, y)$ is called the *Green's function* for the domain Ω. Notice that by the comparison principle, $G(x, y)$ is uniquely defined and in case $u(x)$ is harmonic we get a representation of u in terms of its boundary values only.

Example 6. (The Green's function for a ball) By definition, $G(x, y) = \Gamma(x - y) + v(x)$ and all we need to do is to find $v(x)$ by solving the following problem for a fixed y:

$$\begin{cases} \Delta v(x) = 0, & \text{in } B_R(0) \\ v(x) = -\Gamma(x - y), & \text{on } \partial B_R(0) \end{cases} \tag{38}$$

Now, notice that we simply can't define $v(x) = -\Gamma(x - y)$ everywhere because $v(x)$ would be singular at y, hence can't be harmonic there. So we have to find a way of shifting the singularity to a point outside $B_R(0)$, one way of doing this is by a sort of reflection. Namely, set

$$\overline{y} = \frac{R^2}{|y|^2} y$$

and define

$$v(x) = \begin{cases} -\Gamma(\frac{|y|}{R}(x - \overline{y})), & \text{if } y \neq 0 \\ -\Gamma(R), & \text{if } y = 0 \end{cases}$$

Clearly, this $v(x)$ satisfies 38 and the Green's function for the ball of radius R is

$$G(x, y) = \begin{cases} \Gamma(x - y) - \Gamma(\frac{|y|}{R}(x - \overline{y})), & \text{if } y \neq 0 \\ \Gamma(x) - \Gamma(R), & \text{if } y = 0 \end{cases} \tag{39}$$

Notice that if $u(x) \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is harmonic then using the explicit expression above in 37 we obtain the **Poisson Integral formula**:

$$u(y) = \frac{R^2 - |y|^2}{n\omega_n R} \int_{\partial B_R} \frac{u(x)}{|x - y|^n} \, \mathrm{ds} \tag{40}$$

The above integral depends only on the boundary values, in fact, $u(x)$ as defined above solves the Dirichlet problem for the ball.

Theorem 4.3. *Given $\varphi \in C^0(\partial B_R)$, the function defined by $u(x) = \frac{R^2-|x|^2}{n\omega_n R}\int_{\partial B_R}\frac{\varphi(y)}{|y-x|^n}\,\mathrm{ds}$ is harmonic in B_R, continuous in $\overline{B_R}$ and $u_{|\partial B_R} = \varphi$.*

Proof. We already know that $u(x)$ is harmonic, the only thing left to prove is the continuity. For simplicity, let's set

$$K(x,y) = \frac{R^2-|x|^2}{n\omega_n R|x-y|^n},$$

also called the **Poisson Kernel**.

Notice that by taking $u \equiv 1$ in the Poisson Integral formula, we have that

$$\int_{\partial B_R} K(x,y)\,\mathrm{ds} = 1$$

Now, take $a \in \partial B_R$. Since $\varphi(x)$ is continuous at a, given $\epsilon > 0$ we can find $\delta > 0$ such that $|x-a| < \delta$ implies $|\varphi(x)-\varphi(a)| < \epsilon/2$. Consider now $|x-a| < \frac{\delta}{2}$, we have

$$\begin{aligned}|u(x)-u(a)| &= \left|\int_{\partial B_R} K(x,y)(\varphi(x)-\varphi(a))\,\mathrm{ds}\right| \\ &\leq \int_{|y-a|\leq\delta} K(x,y)|\varphi(x)-\varphi(a)|\,\mathrm{ds} \\ &+ \int_{|y-a|>\delta} K(x,y)|\varphi(x)-\varphi(a)|\,\mathrm{ds} \\ &\leq \frac{\epsilon}{2} + \frac{2\left\|\varphi\right\|_\infty (R^2-|x|^2)R^{n-2}}{(\delta/2)^n}\end{aligned}$$

Since we can choose x very close to a, $|x|^2$ can be as close as we like to R^2. We conclude that $|u(x)-u(a)| < \epsilon$ and $u(x)$ is continuous at a. $\square$

5 The Dirichlet Problem

In this section we generalize the results of the previous section and solve the Dirichlet problem for a general bounded domain Ω. There are different ways of approaching this problem, we will choose the approach that uses the maximum principle as the main tool, the so called **Perron's method**, since it generalizes easily to other types of elliptic equations, most notably to nonlinear ones.

Given a bounded domain Ω and $g \in C^0(\partial\Omega)$, the (classical) Dirichlet problem is to find a function $u(x)$ that solves the following set of conditions

$$\begin{cases}\Delta u(x) = 0 \text{ in } \Omega \\ u \equiv g \text{ on } \partial\Omega.\end{cases} \tag{41}$$

We've already solve the Dirichlet problem in case Ω is a ball (see 4.3). Pictures 7 and 8 illustrates the solution of the classical Dirichlet problem with $\Omega = B_1(0) \subseteq \mathbb{R}^2$, when $g(x,y) = x^2$ and $g(x,y) = xy^2$ respectively. For the general case, we need a couple of definitions first.

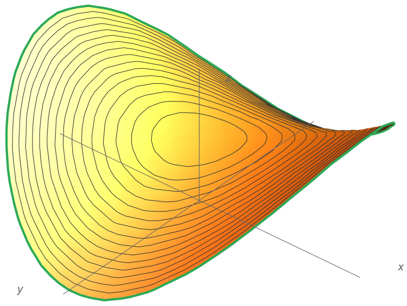

FIGURE 7 Solution with $g(x,y) = x^2$.

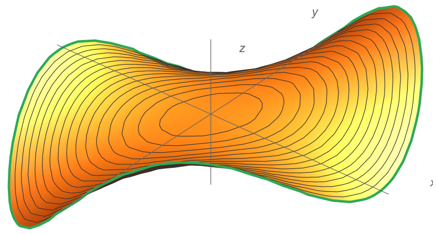

FIGURE 8 Solution with $g(x,y) = xy^2$.

Definition. A continuous function $u(x) \in C^0(\overline{\Omega})$ is called **subharmonic** *(in the viscosity sense)* if for every function $v(x) \in C^2(\Omega)$ such that v touches u from above at x_0 (that is, $v(x_0) = u(x_0)$ and $v \geq u$ in Ω), we have $-\Delta v(x_0) \leq 0$. Analogously, we define **superharmonic** functions as functions that when touched from below by a C^2 function v imply $-\Delta v(x_0) \geq 0$. A function is called **harmonic** if it's both sub- and superharmonic.

It follows directly form the definition that the maximum of subharmonic function is still subharmonic, that is, if $u_1(x), \ldots, u_m(x)$ are subharmonic then

$$u(x) := \max\{u_1(x), \ldots, u_m(x)\}$$

is also subharmonic. What if the sequence of functions u_j is bounded above? Is the sup of subharmonic functions still subharmonic? It turns out that something better is true:

Theorem 5.1. *Given $g(x) \in C^0(\partial\Omega)$, let $S_g := \{u(x)$ subharmonic in $\overline{\Omega} \,|\, u(x) \leq g(x)$ on $\partial\Omega\}$. Set*

$$u(x) := \sup_{v \in S_g} v(x)$$

Then $u(x)$ is harmonic in Ω.

Proof. By definition of sup, if we fix $a \in \Omega$ then $u(a)$ is an accumulation point and as such there is a sequence of functions $v_j(a)$ such that $v_j(a) \to u(a)$. Since $v_j \in S_g$, the sequence v_j is bounded from above, namely by $\sup g$. Without loss of generality, we may also assume that it's also bounded from below (if it's not the case, consider $\max(v_j, \inf g)$) and hence bounded.

Now, choose $R > 0$ such that the ball $B := B_R(a)$ is compactly contained in Ω. Let $V_j(x)$ be the harmonic lifting of $v_j(x)$ in B, that is, the unique harmonic function in B that agrees with $v_j(x)$ on the boundary ∂B and $V_j(x) = v_j(x)$ for $x \in \Omega - B$. Since v_j are subharmonic, V_j are also subharmonic and moreover $V_j \in S_g$.

Notice that, by construction, each V_j is harmonic in B, so by Arzelà-Ascoli theorem, taking a subsequence if necessary, $V_j \to v$ locally uniformly and v is harmonic in B.

By definition, we have $v(a) = u(a)$ and $v(x) \leq u(x)$. Since v is harmonic in B, the proof then follows from the following claim:

Claim. $v(x) \equiv u(x)$ in B.

If the claim is false then there is a point $b \in B$ such that $v(b) < u(b)$, using the definition of sup again, there is a subharmonic function $h \in S_g$ such that $v(b) < h(b)$. Set $w_j := \max\{V_j, h\}$ then, as before, we may assume w_j is bounded with harmonic lifting W_j converging locally uniformly to a harmonic function w. Moreover, $v \le w \le u$ in B and $v(a) = w(a) = u(a)$. By the maximum principle harmonic functions, we have $v = w$ in B, a contradiction since $v(b) < h(b)$. □

Remark. The function $u(x)$ above is sometimes called the **Perron solution**.

The Perron solution is a strong candidate for the solution of the classical Dirichlet Problem: It's harmonic and it's very close to g, by definition. By the comparison principle, if it is a solution, it's unique.

It turns out that it's not always true that the Dirichlet Problem has a solution. In the end, the shape or geometry of $\partial\Omega$ is the deciding factor of whether or not we will have $u(x) = g(x)$ on $\partial\Omega$.

Definition. A point $a \in \partial\Omega$ is called a **regular point** if there is a nonnegative superharmonic function $w(x) \in C^0(\overline{\Omega})$ touching a from above, that is, $w(a) = 0$ and $w > 0$ in $\overline{\Omega} - a$.

We call such $w(x)$, a barrier at $a \in \partial\Omega$. Hence, a point x is regular if there is a barrier at x.

Lemma 5.2. *Let $u(x)$ be the Perron solution and $g \in C^0(\partial\Omega)$ as before. If $a \in \partial\Omega$ is a regular point then $\lim_{x\to a} u(x) = g(a)$.*

Proof. Let $\epsilon > 0$ be given and set $A := \sup|g|$. Since a is regular there is a barrier $w(x)$ touching a from above. By the continuity of g and the definition of barrier, we can find $k > 0$ and $\delta > 0$ such that

$$|g(x) - g(a)| < \epsilon \text{ if } |x - a| < \delta$$

and

$$kw(x) \ge 2M \text{ if } |x - a| \ge \delta$$

The function $g(a) + \epsilon + kw(x) \ge g(x)$ is superharmonic and $g(a) - \epsilon - kw(x) \le g(x)$ is subharmonic, hence using the definition of $u(x)$ we have

$$g(a) - \epsilon - kw(x) \le u(x) \le g(a) + \epsilon + kw(x),$$

which gives $|u(x) - g(a)| \le \epsilon + kw(x)$. Since $kw \to 0$ as $x \to a$, we conclude that $\lim_{x\to a} u(x) = g(a)$. □

We finally have

Theorem 5.3. *The Classical Dirichlet problem is solvable in $\overline{\Omega}$ if and only if every boundary point is regular.*

Proof. If the boundary points are regular, the Perron solution solves the Classical Dirichlet Problem. Conversely, suppose the Classical Dirichlet problem is solvable and let a be a boundary point. It's enough to find a barrier at a. Consider $g(x) = |x - a|$ on $\partial\Omega$, and let $w(x)$ be the harmonic function with boundary values $g(x)$ then $w(x)$ is a barrier at a. □

There are alternatives for explicitly finding a barrier.

Definition. A point $a \in \partial\Omega$ satisfies the **exterior sphere condition** if there is a ball $B_R(b) \subset \Omega$ such that $\overline{B_R(b)} \cap \overline{\Omega} = a$.

Theorem 5.4. *If a point $a \in \partial\Omega$ satisfies the exterior sphere condition then it's regular.*

Proof. Indeed, using the fact that $-|x|$ is superharmonic, we can find $B_R(b) \subset \Omega$ such that

$$w(x) = \begin{cases} R^{n-2} - |x-b|^{2-n} & \text{for } n \geq 3. \\ \log \frac{|x-b|}{R} & \text{for } n = 2. \end{cases}$$

defines a barrier at a. □

Corollary 4. *The Classical Dirichlet problem is solvable for any domain whose boundary is at least C^2.*

Proof. Every boundary $\partial\Omega$ that is at least C^2 satisfies the exterior sphere condition and hence has all of its points regular. This can be seen by using the fact that C^2 functions have a Taylor expansion up to second order and the hessian is diagonalizable by the symmetry of the second derivatives. □

Remark. Notice that the corollary uses the symmetry of the second derivatives which is not necessarily true for C^1 functions. In fact the above is optimal and the Corollary is actually false for C^1 domains in general.